

**Economics Division
University of Southampton
Southampton SO17 1BJ, UK**

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By : Jose Olmo (University of Southampton),

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Nonlinear time varying risk aversion and strategic optimal portfolio allocation

JOSE OLMO*, *University of Southampton*

Abstract

This paper studies the strategic asset allocation problem of individuals with a multiperiod utility function over consumption. Our main contribution is to incorporate the existence of nonlinear dynamics in the relative risk aversion coefficient of power utility functions characterizing individuals' preferences. This coefficient is modeled as a two-regime piecewise linear process not only capturing the evolution of the economy but also nonlinearities due to differences in risk attitudes towards the short and long term defined over the individual's multiperiod, potentially infinite, investment horizon. This modeling strategy is applied in an empirical application to study the impact of model misspecification due to using constant or linear characterizations of relative risk aversion on the optimal portfolio decision of strategic individuals holding a portfolio of stocks, bonds and cash. The empirical results suggest that individual's risk aversion and optimal portfolio allocation do not only vary with the economic environment but are also investment horizon specific.

Keywords: dynamic risk aversion; intertemporal portfolio theory; parametric policy rules; p-value transformation; strategic asset allocation; threshold nonlinearities.

JEL Codes: E32, E52, E62.

*Corresponding Address: School of Social Sciences, Department of Economics, University of Southampton, Room 3015, Bld 58 (Murray Bld), Highfield Campus, Southampton, SO17 1BJ, UK. Phone: (+44) 23 8059 5000. E-mail address: J.B.Olmo@soton.ac.uk.

1 Introduction

Optimal portfolio decisions depend on the details of the economic and financial environment: the financial assets that are available, their expected returns and risks, and the preferences and circumstances of investors. These details become particularly relevant for long-term investors. Such investors must concern themselves not only with expected returns and risks today, but with the way in which expected returns and risks may change over time. It is widely understood at least since the work of Samuelson (1969) and Merton (1969, 1971, 1973) that the solution to a multiperiod portfolio choice problem can be very different from the solution to a static portfolio choice problem. In particular, if investment opportunities vary over time then long-term investors care about shocks to investment opportunities as well as shocks to wealth itself. This can give rise to intertemporal hedging demands for financial assets and lead to strategic asset allocation as a result of the farsighted response of investors to time-varying investment opportunities.

Unfortunately, intertemporal asset allocation models are hard to solve in closed form unless strong assumptions on the investor's objective function such as log preferences or a lognormal distribution for asset returns are imposed. More generally, the lack of closed-form solutions for optimal portfolios with constant relative risk aversion has limited the applicability of the Merton model and has not displaced the Markowitz model. This situation has begun to change as a result of several developments in numerical methods and continuous time finance models such as Balduzzi and Lynch (1999), Barberis (2000), Brennan et al. (1997, 1999) and Lynch (2001). Approximate analytical solutions to the Merton model have been developed in Campbell and Viceira (1999, 2001, 2002) and Campbell et al. (2003) for models exhibiting an intertemporal elasticity of substitution close to one. A recent alternative to solving the investor optimal portfolio problem has been proposed by Brandt (1999), Ait-Sahalia and Brandt (2001) and Brandt and Santa-Clara (2006). These authors focus directly on the dependence of the portfolio

weights on the predictors. They do this by solving sample analogues of the conditional Euler equations that characterize the optimal portfolio choice. These seminal contributions to the literature on optimal asset allocation impose in most cases an exogenous and constant relative risk aversion coefficient to model individuals' risk attitude and derive the corresponding optimal portfolio allocation under different assumptions on such relative risk aversion coefficient.

The assumption that the relative risk aversion coefficient is constant over time can be misleading in many circumstances and have important implications on the asset allocation of strategic investors if the utility function used to describe investor's preferences is misspecified. This observation is supported by some empirical evidence, see Cohn et al. (1975) and Friend and Blume (1975), that document that the fraction of wealth households invest in stocks increases with their wealth. One common explanation for the observed pattern of portfolio shares is that relative risk aversion decreases with wealth. Morin and Suarez (1983), for example, corroborate this finding empirically using portfolio data to elicit households' preferences. A byproduct of this observation is that relative risk aversion is dynamic unless the wealth held by the investor is constant over time.

Another plausible explanation for the existence of dynamics in the relative risk aversion coefficient is the presence of habit formation. In a habit formation model, see Campbell and Cochrane (1999), Chan and Kogan (2002), and more recently, Brandt and Wang (2003), the representative agent's relative risk aversion varies with the difference between consumption and the agent's habit. This habit can be interpreted as a minimum subsistence level required by the individual or some dynamic value that is formed through past consumption. The existence of dynamics in relative risk aversion can also be captured by changes in the risk aversion coefficient of individuals. A more uncertain economic environment can lead individuals to consider more cautiously the same investment opportunities than under a favourable environment. The plausibility of a time-varying risk aversion coefficient gains importance for investors with mul-

tiperiod objective functions. For these individuals the influence on their investment decision of variation in future risk aversion is twofold. First, relative risk aversion can change due to the future evolution of the economy, and second, strategic investors can have different risk attitudes towards the short and long term with these two terms defined over the individual's multiperiod investment horizon.

The aim of this paper is to incorporate these features into standard formulations of the strategic optimal portfolio allocation problem and empirically assess the effect on optimal portfolio allocation of misspecification of the utility function due to assuming wrong functional forms for the relative risk aversion coefficient. To do this we entertain individuals characterized by three distinguishing features. First, these individuals are considered to invest strategically, see Brennan et al. (1997), in the sense that their investment decisions entertain more than one period. Second, these individuals exhibit a risk aversion coefficient that is time varying with dynamics driven by a set of state variables reflecting macroeconomic and financial conditions. Finally, these individuals react differently to the short term than the long term defined over their multiperiod utility functions. This assumption is reflected in a risk aversion coefficient that not only changes with time but also with the investment horizon. In order to model these two sources of variation in risk aversion we consider a piecewise linear model with threshold nonlinearity determined by the period within the individual's investment horizon that separates the short from the long term. Intuitively, the individual exhibits different risk attitudes to negative events taking place before the threshold period than in the distant future. The choice of a piecewise linear function is for modeling convenience and simplicity. Nevertheless, a piecewise linear function can be considered as a general alternative to a linear function, that provides a better approximation to a smooth nonlinear function than that from a (global) linear function.

As mentioned above the existence of a multiperiod optimal decision problem implies in most cases the lack of closed form solutions and the need of dynamic stochastic programming

methods. To overcome this problem we take advantage of the methodology proposed in Aït-Sahalia and Brandt (2001) and assume that the optimal portfolio weights are driven by a parametric linear policy rule. In this way we entertain two different parametric policy rules: a portfolio policy rule driving the dynamics of the optimal asset allocation in terms of a set of state variables, and a risk aversion policy rule driving the dynamics of relative risk aversion. By doing this we show that the first order conditions of the optimization problem over a potentially infinite investment horizon can be expressed in terms of a simple system of equations that is overidentified. The overidentification property entails a natural empirical representation of the system of equations that can be used for parameter estimation and statistical inference. More specifically, we can estimate the marginal contributions of the state variables to the parametric policy rules using the sample counterparts of the multiple Euler equations that characterize the optimal portfolio choice and test the correct specification of the parametric policy rules using a version of the overidentification J-test developed by Hansen (1982).

The choice of a piecewise linear function for modeling the dynamics of risk aversion is formalized by proposing a likelihood ratio test comparing the suitability of the linear and nonlinear risk aversion specifications. The econometric methodology is similar in spirit to the seminal papers by Andrews (1993), Andrews and Ploberger (1994) and Hansen (1996) that discuss how to make inference when a nuisance parameter is not identified under the null hypothesis. In our setting the nuisance parameter that is not identified under the null hypothesis is the threshold value, that is, the period in the individual's strategic horizon separating the short term from the long term. The asymptotic distribution of this test is nonstandard and is approximated using a p-value transformation implemented through a multiplier method applied to the first order conditions of the individual's maximization problem.

This methodology is explored in an empirical application assessing the optimal portfolio decisions of an strategic investor holding a tactical portfolio given by stocks, bonds and cash

spanning thirty years of financial returns. This empirical exercise closely follows similar studies such as Brennan et al. (1997), Brandt (1999) and Campbell et al. (2003), among many others. The investor is assumed to invest in three assets - a one-month Treasury bill as riskless security, a long-term bond, and an equity portfolio. We consider a set of state variables that is common in the predictive literature on asset pricing and portfolio theory: the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of excess stock and bond returns. The effect of the state variables is not uniform across assets, in fact, we observe more predictive power for the optimal allocation to bonds than stocks. These state variables also help to predict the dynamics of the individual's relative risk aversion coefficient. Interestingly, we find empirical support for the existence of a short term and a long term regime in such coefficient. Our results are robust to the choice of strategic horizon and to replacing consumption for wealth in the individual's utility function.

This paper is related to several literatures on optimal portfolio allocation. The closest contributions are the sequel of papers by Brandt (1999), Aït-Sahalia and Brandt (2001) and Brandt and Santa-Clara (2006). It is also closely related to the literature on strategic asset allocation initiated by Merton (1968, 1971, 1973) and continued by Campbell and Viceira (1999, 2011, 2002). It connects with the literature on habit formation widely explored in studies of stock market behavior by Campbell and Cochrane (1999), Chan and Kogan (2002) or Brunnermeier and Nagel (2008), among many others. Our approach to modeling the dynamics of risk aversion shares the view introduced by Brandt and Wang (2003). These authors formulate a consumption-based asset pricing model in which aggregate risk aversion varies in response to news about consumption growth and inflation dynamics. In contrast to previous models, they explicitly model the dynamics of risk aversion using a stochastic mean-reverting autoregressive model. Finally, the current paper is also related to the literature on nonlinearity tests and inference under the presence of nuisance parameters such as Andrews (1993), Andrews and

Ploberger (1994) and Hansen (1996).

The rest of the article is structured as follows. Section 2 presents the model and derives the system of overidentified equations corresponding to the first order conditions of the multiperiod maximization problem of an individual exhibiting a time-varying piecewise linear risk aversion coefficient. Section 3 discusses the implementation of the generalized method of moments (GMM) to estimate the optimal portfolio weights and the risk aversion coefficients and briefly discusses the corresponding asymptotic theory. Section 4 presents two types of econometric tests to assess the parametric assumptions used in the development of our model. First, we introduce in detail a threshold nonlinearity test to assess statistically the existence of piecewise nonlinearities in the individual's strategic multiperiod utility function, and second, we discuss several specification tests to assess the suitability of the parametric policy rules proposed in the paper. Section 5 presents an empirical application to derive the optimal allocation to a portfolio of stocks, bonds and cash for a strategic investor with a multiperiod utility function. Section 6 concludes.

2 The Model

Consider the portfolio choice of an investor who maximizes the expected utility of real consumption (c_t) over multiple, potentially infinite, K periods. Assume that the utility function is additively time separable and takes the form

$$\sum_{j=0}^K \beta^j E_t \left[\frac{c_{t+j}^{1-\gamma(j)}}{1-\gamma(j)} \right], \quad (1)$$

with

$$\ln \gamma(j) \equiv \ln \gamma(j, z_{t+j}) = \gamma' z_{t+j} + \eta' z_{t+j} 1(j > k_0) \quad (2)$$

where z_{t+j} is a $n \times 1$ vector comprising a constant and a set of $n - 1$ macroeconomic and financial variables reflecting all the information available to the investor at time $t + j$. Two parameters describe individuals' preferences: the discount factor β measures patience, the willingness to give up consumption today for consumption tomorrow, and the coefficient $\gamma(j)$ captures risk aversion, the reluctance to trade consumption for a fair gamble over consumption today. The parameters $\gamma = (\gamma_c, \gamma_1, \dots, \gamma_{n-1})'$ and $\eta = (\eta_c, \eta_1, \dots, \eta_{n-1})'$ capture the effect of these variables on the risk aversion coefficient. The dynamics of the risk aversion function are driven by changes in macroeconomic conditions and the individual's risk attitude with respect to the strategic investment horizon. The threshold k_0 denotes the period separating the short from the long term defined over the K -period individual's investment horizon, and the vector η captures the differences in the risk aversion coefficient between the short and long term. The structural model (2) for the risk aversion coefficient $\gamma(j)$ can be interpreted as an alternative to the stochastic mean-reverting autoregressive process proposed in Brandt and Wang (2003) for modeling relative risk aversion. The above function can be alternatively expressed as

$$\ln \gamma(j) = \gamma' z_{t+j} + \eta' z_{t+j} \mathbf{1}(\omega_j > \omega_0) \quad (3)$$

with $\omega_0, \omega_j \in [\omega_{\min}, \omega_{\max}] \in (0, 1)$ and such that $k_0 = \lceil \omega_0 K \rceil$ where $\lceil \cdot \rceil$ denotes the integer part of the value inside the brackets.

The individual begins life with an exogenous endowment $w_0 \geq 0$. This endowment accumulates over time according to the equation

$$w_{t+1} = (1 + r_{t+1}^p)(w_t - c_t). \quad (4)$$

At the beginning of the period $t + 1$ the individual receives income from allocating resources in

an investment portfolio offering a real return r_{t+1}^p . The portfolio return is defined as

$$r_{t+1}^p(\alpha_t) = r_{f,t+1} + \alpha_t' r_{t+1}^e, \quad (5)$$

with $r_{t+1}^e = (r_{1,t+1} - r_{f,t+1}, \dots, r_{m,t+1} - r_{f,t+1})'$ denoting the vector of excess returns on the m risky assets over the real risk-free rate $r_{f,t+1}$, and $\alpha_t = (\alpha_{1,t}, \dots, \alpha_{m,t})'$ denoting the different allocations to risky assets. In order to be able to solve a multiperiod maximization problem that accommodates in a parsimonious way arbitrarily long investment horizons, we entertain the parametric portfolio policy rule introduced in the seminal contributions of Aït-Sahalia and Brandt (2001), Brandt and Santa-Clara (2006) and Brandt et al. (2009):

$$\alpha_{h,t+i} = \lambda_h' z_{t+i}, \quad h = 1, \dots, m, \quad (6)$$

with $z_t = (1, z_{1,t}, \dots, z_{n,t})'$ a set of state variables describing the evolution of the economy and reflecting all the relevant information available to the individual at time t , and $\lambda_h = (\lambda_{h,1}, \dots, \lambda_{h,n})'$ the corresponding vector of parameters. Time variation of the optimal asset allocation is introduced through the dynamics of the state variables. This specification of the portfolio weights has two main features. First, it allows us to study the marginal effects of the state variables on the optimal portfolio weights through the set of parameters λ , and second, it avoids the introduction of time consuming stochastic dynamic programming methods. As a byproduct, this specification of the optimal portfolio weights could accommodate arbitrarily long horizons in the individual's objective function. A potential downside of this parametric approach is to force the individual's optimal portfolio policy rule to be linear and with the same parameter values over the long term horizon. Nevertheless, for finite horizon ($K < \infty$) objective functions, more sophisticated models can be developed that entertain different parametric portfolio policy rules for different investment horizons $i = 1, \dots, K$. This approach

significantly increases the computational complexity of the methodology and is beyond the aim of this study.

The objective of solving the asset allocation problem for long term investors is challenged by the definition of the individuals' preferences in terms of consumption rather than wealth. We, nevertheless, pursue this route as it leads to more realistic representations of the preferences of long term individuals. The augmented complexity of the optimization problem introduced by jointly maximizing over the optimal portfolio weights and consumption is reduced in our framework by assuming that consumption is simply a fixed fraction of wealth w_t in each period:

$$c_t = (1 - \theta)w_t \text{ with } 0 < \theta \leq 1. \quad (7)$$

This assumption simplifies considerably the optimization problem but allows, at the same time, for some flexibility in the study of the optimal portfolio allocation problem under different consumption streams. The limiting case $\theta = 1$ describes, for example, the strategic asset allocation of a long-term investor only interested with maximizing the utility of the stream of wealth rather than consumption. This assumption, although restrictive from a macroeconomic perspective, can be motivated under different interpretations of the optimal consumption-wealth ratio. Early pioneering studies such as Friedman (1957) already notes this relation between consumption and wealth in his celebrated Permanent Income Hypothesis. Samuelson (1969) also observes such relationship between consumption and wealth in a Ramsey model of consumption. In Samuelson's seminal study the fraction of wealth consumed by the individual in each period differs depending on whether utility is of log form or reflects risk aversion. In the former case, for example, the optimal consumption-wealth ratio is characterized by the time preference parameter β used by the individual for discounting future utility. Recently, He and Krishnamurthy (2013), in an asset pricing context, also assume a constant relationship between

consumption and wealth over time.

Condition (7) allows us to write the investor's wealth process at time $t + j$ in terms of the compound j -period gross return, the parameter θ and w_t . More formally,

$$w_{t+j} = \theta^j \prod_{i=1}^j (1 + r_{t+i}^p(\lambda'_h z_{t+i-1})) w_t. \quad (8)$$

This representation of the individual's stream of wealth over time has major implications for parameter estimation and statistical inference.

2.1 Optimal portfolio choice under risk aversion

In this section we derive the first order conditions of the long term optimal portfolio choice problem for a risk-averse individual with preferences described above. Expression (8) and simple algebra shows that the individual's maximization problem can be written as

$$\max_{\{\lambda_{hs}\}} \left\{ \sum_{j=0}^K E_t \left[\delta_{t,j}^j \frac{c_t^{1-\gamma(j)}}{1-\gamma(j)} \left(\prod_{i=1}^j (1 + r_{t+i}^p(\lambda'_h z_{t+i-1})) \right)^{1-\gamma(j)} \right] \right\} \quad (9)$$

with $\delta_{t,j} = \beta \theta^{1-\gamma(j)}$. Under the initial condition $c_t = 1$ the first order conditions of this optimization problem with respect to the vector of parameters λ_{hs} , with $h = 1, \dots, m$ and $s = 1, \dots, n$, provide for each $\omega \in [\omega_{\min}, \omega_{\max}]$ a system of mn equations characterized by the following conditions:

$$E_t \left[\sum_{j=1}^K \delta_{t,j}^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) \right] = 0 \quad (10)$$

with

$$\psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) = \left(\sum_{i=1}^j \frac{z_{s,t+i-1} r_{h,t+i}^e}{1 + r_{t+i}^p(\lambda'_h z_{t+i-1})} \right) \left(\prod_{i=1}^j (1 + r_{t+i}^p(\lambda'_h z_{t+i-1})) \right)^{1-\gamma(j)}. \quad (11)$$

The set of conditional moments (10) can be expressed as an augmented set of unconditional moments if we assume that the conditioning information set can be reflected by the set of state variables z_t . Then, the set of unconditional moments is

$$E \left[\sum_{j=1}^K \delta_{t,j}^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) \otimes z_t \right] = 0 \quad (12)$$

where \otimes denotes element by element multiplication. More specifically, expression (12) yields the following system of mn^2 conditions:

$$\phi_{h,s}^{\tilde{s}}(\mu, \omega) \equiv E \left[\sum_{j=1}^K \delta_{t,j}^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) z_{\tilde{s},t} \right] = 0, \quad (13)$$

where $\mu = (\lambda, \gamma, \eta)$ and $h = 1, \dots, m$, $s, \tilde{s} = 1, \dots, n$ and $z_{1,t} = 1$.

The main advantage of this approach is that the first order conditions of the maximization problem of a strategic investor with power utility yield a simple system of equations that is overidentified and provides a very intuitive empirical representation. This property is exploited in the econometric section to derive suitable estimators of the portfolio weights and carry out statistical tests of the specifications (3) and (6).

3 Econometric methods: estimation

This section presents suitable methods to estimate the optimal portfolio weights and the parameters driving the dynamics of the risk aversion coefficient. A suitable empirical representation of the Euler equation (13) is

$$\widehat{\phi}_{h,s}^{\tilde{s}}(\mu, \omega) \equiv \frac{1}{T - K^*} \sum_{t=1}^{T-K^*} e_{hs,t}(\mu, \omega) z_{\tilde{s},t} = 0 \quad (14)$$

with

$$e_{hs,t}(\mu, \omega) = \sum_{j=1}^{K^*} \delta_{t,j}^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) \quad (15)$$

and T is the sample size used for estimating the model parameters; $K^* = K$ for the finite multiperiod case and $K^* = \min\{j \mid \beta^j \leq \text{tol}, j = 1, \dots, \infty\}$ for the infinite horizon case; tol denotes a tolerance level determining the truncation of the infinite horizon model. Statistical tests can be devised to assess the suitability of the finite truncation K^* in infinite horizon models.

For each $\omega \in [\omega_{\min}, \omega_{\max}]$, let $g(\mu, \omega)$ and $g_T(\mu, \omega)$ be the $mn^2 \times 1$ vectors that stack each of the sample moments $\phi_{h,s}^{\tilde{s}}(\mu, \omega)$ and $\hat{\phi}_{h,s}^{\tilde{s}}(\mu, \omega)$, respectively, indexed by h, s and \tilde{s} , with $h = 1, \dots, m$ and $s, \tilde{s} = 1, \dots, n$. The idea behind GMM is to choose $\hat{\mu}_T$ so as to make the sample moments $g_T(\mu, \omega)$ as close to zero as possible. An important distinction with respect to the linear case is the existence of a threshold parameter ω that determines the presence of nonlinearities in the investor's strategic behavior. This parameter introduces a break in the individual's objective function that determines two regimes in the functional form of the risk aversion coefficient.

To estimate the model parameters in the general case given by absence of knowledge of the true population parameter ω , we propose a two-step estimation procedure. First, for each ω , we define the set of parameter estimators $\hat{\mu}_T(\omega)$ of the true parameter vector $\mu \in \Theta$ as

$$\hat{\mu}_T(\omega) = \arg \min_{\mu \in \Theta} g'_T(\mu, \omega) V_T^{-1}(\omega) g_T(\mu, \omega) \quad (16)$$

where

$$V_T(\omega) = \frac{1}{T - K^*} \sum_{t=1}^{T-K^*} e_{h_1 s_1, t}(\mu, \omega) e_{h_2 s_2, t}(\mu, \omega) z_{\tilde{s}_1, t} z_{\tilde{s}_2, t}$$

$$+ \frac{1}{T - K^*} \sum_{t=1}^{T-K^*} \sum_{\substack{t'=1 \\ t' \neq t}}^{T-K^*} e_{h_1 s_1, t}(\mu, \omega) e_{h_2 s_2, t'}(\mu, \omega) z_{\tilde{s}_1, t} z_{\tilde{s}_2, t'} \quad (17)$$

is a consistent estimator of $V_0(\omega) = E[g_T(\mu, \omega)g_T'(\mu, \omega)]$, a $mn^2 \times mn^2$, possibly random, non-negative definite weight matrix, whose rank is greater than or equal to mn . This estimator highlights the strong persistence in the covariance matrix $V_0(\omega)$. This persistence is due to the presence of serial correlation produced by considering a strategic investment horizon ($K^* > 1$) in the individual's objective function. The second step of the estimation process consists of finding the strategic horizon that minimizes the objective function on ω . More formally,

$$\widehat{\omega}_T = \arg \min_{\omega \in [\omega_{\min}, \omega_{\max}]} \widehat{\mu}_T(\omega). \quad (18)$$

The strategic horizon associated to the optimal $\widehat{\omega}_T$ is given by $\widehat{k}_T = \lceil \widehat{\omega}_T K \rceil$. In order to guarantee the consistency of the estimators of the model parameters we need to assume that $\{r_{t+1}^e, z_t\}$ is strictly stationary and α -mixing with α of size $-r/(r-2)$, with $r > 2$; $E[z_t z_t']$ is nonsingular and there exists some $\delta > 0$ such that $E[||z_t||^{2r+\delta}] < \infty$, see proposition 1 of Giacomini and Komunjer (2005) for similar assumptions in a quantile regression setting.

A similar two-step procedure is proposed by Seo and Shin (2014) in a dynamic panel data setting. In this setting these authors derive the asymptotic distribution of the model parameters including the threshold. In contrast to the conventional theory for threshold estimators derived from least squares, e.g. Chan (1993) and Hansen (1996), the threshold parameter estimator is asymptotically normal irrespective of whether the regression function is continuous or not. Furthermore, these authors show that the standard inference on the threshold estimator is feasible, though the convergence rate is slower than \sqrt{T} . A direct application of the asymptotic results obtained in Seo and Shin (2014) to this setting entails that under some further regularity

conditions, as for example, $K/T \rightarrow 0$, it holds that

$$\sqrt{T-K} (\widehat{\mu}_T(\widehat{\omega}_T) - \mu) \xrightarrow{d} N\left(0, (D'(\omega)\Omega^{-1}(\omega)D(\omega))^{-1}\right) \quad (19)$$

with $\Omega(\omega) = E[g(\mu, \omega)g'(\mu, \omega)]$ and $D(\omega) \equiv D(\mu, \omega) = \frac{\partial g(\mu, \omega)}{\partial \mu}$ a function that is continuous in the vector μ .

4 Econometric methods: hypothesis testing

This section presents a threshold nonlinearity test to statistically assess whether there exist dynamics in the risk aversion coefficient that can be modeled as a two-regime piecewise linear process. Second, we exploit the overidentified system of equations (12) to propose a specification test for the parametric formulation of the risk aversion function (3) and the policy rule (6).

4.1 Threshold nonlinearity tests

Following the literature on threshold and structural break models we will distinguish two cases. One, in which the timing of the break ω_0 is known, and a second case, in which ω_0 is not identified under the null hypothesis. In both scenarios the null hypothesis corresponds to the case

$$H_0 : \eta_c = \eta_1 = \dots = \eta_{n-1} = 0 \text{ against } H_A : \eta_s \neq 0 \text{ for some } s = c, 1, \dots, n-1,$$

in the dynamic risk aversion coefficient (3). This composite test is standard for ω_0 known and appropriate test statistics can be deployed by exploiting the overidentified system of equations (12). More specifically, a suitable nonlinearity test for the null hypothesis is the likelihood ratio

test

$$L_{K^*}(\omega_0) = (T - K^*) (s(\widehat{\mu}_{0T}, \omega_0) - s(\widehat{\mu}_T, \omega_0)) \quad (20)$$

with $s(\widehat{\mu}_T, \omega_0) = g'_T(\widehat{\mu}_T, \omega_0) \widehat{V}_T^{-1}(\omega_0) g_T(\widehat{\mu}_T, \omega_0)$. Similarly, $s(\widehat{\mu}_{0T}, \omega_0)$ is the version of the statistic under the null hypothesis H_0 . It is important to note that the covariance matrix of the parameter estimators refers for both statistics $s(\widehat{\mu}_T, \omega_0)$ and $s(\widehat{\mu}_{0T}, \omega_0)$ to the same consistent estimator of the covariance matrix $V_0(\omega_0)$ estimated under the unrestricted model. A natural candidate robust to the presence of serial correlation in the sample moments is the sample covariance matrix

$$\widehat{V}_T(\omega) = \frac{1}{T - K^*} \sum_{t=1}^{T-K^*} \widehat{e}_{h_1 s_1, t} \widehat{e}_{h_2 s_2, t} z_{\widetilde{s}_1, t} z_{\widetilde{s}_2, t} + \frac{1}{T - K^*} \sum_{t=1}^{T-K^*} \sum_{\substack{t'=1 \\ t' \neq t}}^{T-K^*} \widehat{e}_{h_1 s_1, t} \widehat{e}_{h_2 s_2, t'} z_{\widetilde{s}_1, t} z_{\widetilde{s}_2, t'} \quad (21)$$

with

$$\widehat{e}_{hs, t} = \sum_{j=1}^{K^*} \widehat{\delta}_{t, j}^j \psi_{t, j}(z_s; \widehat{\lambda}_h, \widehat{\gamma}_T, \widehat{\eta}_T, \omega) \quad (22)$$

where $\widehat{\delta}_{t, j} = \beta \theta^{1-\widehat{\gamma}(j)}$, $\widehat{\gamma}(j) = \exp(\widehat{\gamma}'_T z_{t+j} + \widehat{\eta}'_T z_{t+j} 1(\omega_j > \omega))$ and

$$\psi_{t, j}(z_s; \widehat{\lambda}_h, \widehat{\gamma}_T, \widehat{\eta}_T, \omega) = \left(\sum_{i=1}^j \frac{z_{s, t+i-1} r_{h, t+i}^e}{1 + r_{t+i}^p (\widehat{\lambda}'_h z_{t+i-1})} \right) \left(\prod_{i=1}^j (1 + r_{t+i}^p (\widehat{\lambda}'_h z_{t+i-1})) \right)^{1-\widehat{\gamma}(j)}.$$

Under these conditions, it holds that

$$L_{K^*}(\omega) \xrightarrow{d} \chi_n^2 \quad (23)$$

with n the number of restrictions implied by the null hypothesis H_0 .

A similar testing procedure can be developed to assess the existence of linear dynamics in the risk aversion coefficient against constant risk aversion. To do this, we take as benchmark

under the alternative hypothesis a simplified version of (3) given by $\gamma(j) = \exp(\gamma' z_{t+j})$. The relevant hypothesis is

$$H_0 : \gamma_1 = \dots = \gamma_{n-1} = 0 \text{ against } H_A : \gamma_s \neq 0 \text{ for some } s = 1, \dots, n-1, \quad (24)$$

with the vector $(\gamma_1, \dots, \gamma_{n-1})'$ denoting the parameters associated to the state variables $z_{2,t}, \dots, z_{n,t}$.

For the most interesting cases, such as testing for nonlinearity of the preferences of the strategic investor when ω_0 is not known, $\omega_0 \in [\omega_{\min}, \omega_{\max}]$ is a nuisance parameter that cannot be identified under the null hypothesis. In this case Hansen (1996) shows that the composite nonlinearity test is nonstandard. As proposed by this author, see also Davies (1977, 1987) or Andrews and Ploberger (1994) in different contexts, hypothesis tests for nonlinearity can be based on different functionals of the relevant test statistic computed over the domain of the nuisance parameter. In our framework the relevant test statistic is $l_{K^*} = \sup_{\omega \in [\omega_{\min}, \omega_{\max}]} L_{K^*}(\omega)$ with sup standing for the supremum functional. In this case the statistic $s(\mu, \omega)$ is a function on $\omega \in [\omega_{\min}, \omega_{\max}]$. To formalize the asymptotic distribution of $L_{K^*}(\omega)$ we define the covariance function

$$\Sigma_0(\omega_1, \omega_2) = E [g_T(\mu, \omega_1) g_T'(\mu, \omega_2)] \quad (25)$$

and its empirical counterpart

$$\begin{aligned} \widehat{\Sigma}_T(\omega_1, \omega_2) &= \frac{1}{T - K^*} \sum_{t=1}^{T-K^*} \widehat{e}_{h_1 s_1, t}(\widehat{\mu}_T, \omega_1) \widehat{e}_{h_2 s_2, t}(\widehat{\mu}_T, \omega_2) z_{\widetilde{s}_1, t} z_{\widetilde{s}_2, t} \\ &+ \frac{1}{T - K^*} \sum_{t=1}^{T-K^*} \sum_{\substack{t'=1 \\ t' \neq t}}^{T-K^*} \widehat{e}_{h_1 s_1, t}(\widehat{\mu}_T, \omega_1) \widehat{e}_{h_2 s_2, t'}(\widehat{\mu}_T, \omega_2) z_{\widetilde{s}_1, t} z_{\widetilde{s}_2, t'} \end{aligned}$$

with $\omega_1, \omega_2 \in [\omega_{\min}, \omega_{\max}]$. These expressions are the functional counterparts of the covariance matrices $V_0(\omega)$ and $\widehat{V}_T(\omega)$, respectively.

To derive the asymptotic distribution of the relevant test we define the processes $S_T(\widehat{\mu}_T, \omega) = \sqrt{T - K^*} g_T(\widehat{\mu}_T, \omega)$ and $S_{0T}(\widehat{\mu}_{0T}, \omega) = \sqrt{T - K^*} g_T(\widehat{\mu}_{0T}, \omega)$. Under some suitable regularity conditions on the uniform convergence of $\widehat{\Sigma}_T(\omega_1, \omega_2)$ to $\Sigma_0(\omega_1, \omega_2)$ over its compact support, see Hansen (1996) for more technical details, the process $S_T(\widehat{\mu}_T, \omega)$ converges weakly to a multivariate zero mean Gaussian process, $S(\mu, \omega)$, defined by the covariance function $\Sigma_0(\omega_1, \omega_2)$. Similarly, under the null hypothesis H_0 the process $S_{0T}(\widehat{\mu}_{0T}, \omega)$ converges to a multivariate zero-mean Gaussian process $S_0(\mu_0, \omega)$. Therefore, under the null hypothesis, the process $L_{K^*}(\omega)$ converges weakly to the following chi-square process

$$L_0(\omega) = S'_0(\mu_0, \omega)\Sigma_0(\omega, \omega)^{-1}S_0(\mu_0, \omega) - S'(\mu, \omega)\Sigma_0(\omega, \omega)^{-1}S(\mu, \omega). \quad (26)$$

Consequently, the asymptotic distribution of the supremum functional is $l_0 = \sup_{\omega \in [\omega_{\min}, \omega_{\max}]} L_0(\omega)$. Since the null distribution (26) depends upon the covariance function Σ_0 , critical values cannot be tabulated. To obtain the p -values of the test we derive a p -value transformation similar in spirit to the work of Hansen (1996).

Let $F_0(\cdot)$ denote the distribution function of l_0 , and define $p_T = 1 - F_0(l_{K^*})$. The above result shows that p_T converges in probability to $p_0 = 1 - F_0(l_0)$, that under the null hypothesis is uniform on $[0, 1]$. Thus the asymptotic null distribution of p_T is free of nuisance parameters. The rejection rule of our test is given by $p_T < \alpha$ with α the significance level and p_T the asymptotic p -value. The random variable l_0 can be written as a continuous functional of the Gaussian processes $S(\mu, \omega)$ and $S_0(\mu_0, \omega)$, which are completely described by the covariance function $\Sigma_0(\omega_1, \omega_2)$. To implement the p -value transformation, we operate conditional on the sample $\mathfrak{S} = \{(r'_{t+1}, z'_t)\}_{t=1}^T$ and define the conditional multivariate mean-zero Gaussian processes \widehat{S}_T and \widehat{S}_{0T} . These processes can be generated by letting $\{v_t\}_{t=0}^{T-1}$ be *i.i.d.* $N(0, 1)$ random variables,

and setting for $h = 1, \dots, m$ and $s, \tilde{s} = 1, \dots, n$, the following processes

$$\widehat{S}_T(\widehat{\mu}_T, \omega) = \frac{1}{\sqrt{T - K^*}} \sum_{t=1}^{T-K^*} \widehat{e}_{hs,t} z_{\tilde{s},t} v_t. \quad (27)$$

Similarly, we have

$$\widehat{S}_{0T}(\widehat{\mu}_{0T}, \omega) = \frac{1}{\sqrt{T - K^*}} \sum_{t=1}^{T-K^*} \widehat{e}_{hs,t}^0 z_{\tilde{s},t} v_t \quad (28)$$

with $\widehat{e}_{hs,t}^0$ the version of $\widehat{e}_{hs,t}$ in (22) obtained under the null hypothesis H_0 . The corresponding conditional chi-square process is

$$\widehat{L}_{K^*}(\omega) = \widehat{S}_{0T}'(\widehat{\mu}_{0T}, \omega) \widehat{V}_T^{-1}(\omega) \widehat{S}_{0T}(\widehat{\mu}_{0T}, \omega) - \widehat{S}_T'(\widehat{\mu}_T, \omega) \widehat{V}_T^{-1}(\omega) \widehat{S}_T(\widehat{\mu}_T, \omega) \quad (29)$$

and the corresponding test statistic is $\widehat{l}_{K^*} = \sup_{\omega \in [\omega_{\min}, \omega_{\max}]} \widehat{L}_{K^*}(\omega)$. Finally, let \widehat{F}_0 denote the conditional distribution function of \widehat{l}_{K^*} and $\widehat{p}_T = 1 - \widehat{F}_0(\widehat{l}_{K^*})$.

The introduction of the zero-mean random variable v_t implies that, conditionally, the covariance function of $\widehat{S}_T(\widehat{\mu}_T, \omega)$ is equal to $\widehat{\Sigma}_T(\omega, \omega)$, that is,

$$E \left[\frac{1}{T - K^*} \sum_{t=1}^{T-K^*} \sum_{t'=1}^{T-K^*} \widehat{e}_{hs,t} z_{\tilde{s},t} \widehat{e}_{hs,t'} z_{\tilde{s},t'} v_t v_{t'} \mid \mathfrak{S} \right] = \frac{1}{T - K^*} \sum_{t=1}^{T-K^*} \sum_{t'=1}^{T-K^*} \widehat{e}_{hs,t} z_{\tilde{s},t} \widehat{e}_{hs,t'} z_{\tilde{s},t'} E[v_t v_{t'} \mid \mathfrak{S}] = \widehat{\Sigma}_T(\omega, \omega).$$

Following similar arguments to the proof of Theorem 2 in Hansen (1996), it can be shown that the quantity \widehat{p}_T is asymptotically equivalent to p_T under both the null and alternative hypotheses. The conditional distribution function \widehat{F}_T is not directly observable so neither is the random variable \widehat{p}_T . Nevertheless, these quantities can be approximated to any desired degree of accuracy using standard simulation techniques. The following algorithm shows the implementation of this p-value transformation. Let Ω_N define a grid of N points over the compact set $[\omega_{\min}, \omega_{\max}]$, and let ω_i for $i = 1, \dots, N$ be the set of equidistant points in such

grid with $\omega_1 = \omega_{\min}$ and $\omega_N = \omega_{\max}$; for $j = 1, \dots, J$, execute the following steps:

- i) generate the sequence $\{v_{jt}\}_{t=1}^T$ *i.i.d.* random variables;
- ii) conditional on the sample $\mathfrak{S} = \{(r'_{t+1}, z'_t)'\}_{t=1}^T$, set the quantities $\widehat{S}_T^j(\widehat{\mu}_T, \omega_i)$ and $\widehat{S}_{0T}^j(\widehat{\mu}_{0T}, \omega_i)$;
- iii) set $\widehat{L}_{K^*}^j(\omega_i) = \widehat{S}_{0T}^j(\widehat{\mu}_{0T}, \omega_i) \widehat{V}_T^{-1}(\omega_i) \widehat{S}_{0T}^j(\widehat{\mu}_{0T}, \omega_i) - \widehat{S}_T^j(\widehat{\mu}_T, \omega_i) \widehat{V}_T^{-1}(\omega_i) \widehat{S}_T^j(\widehat{\mu}_T, \omega_i)$;
- iv) set $\widehat{l}_{K^*}^j = \sup_{\omega \in \Omega_N} \widehat{L}_{K^*}^j(\omega_i)$.

This gives a random sample $(\widehat{l}_{K^*}^1, \dots, \widehat{l}_{K^*}^J)$ from the conditional distribution \widehat{F}_T . The percentage of these artificial observations which exceeds the actual test statistic l_{K^*} : $\widehat{p}_T^J = \frac{1}{J} \sum_{j=1}^J 1(\widehat{l}_{K^*}^j > l_{K^*})$ is according to the Glivenko-Cantelli theorem a consistent approximation of \widehat{p}_T as $J \rightarrow \infty$. In practice, the null hypothesis H_0 is rejected if $\widehat{p}_T < \alpha$.

4.2 Specification tests

This section discusses a second type of tests to assess the correct specification of the functional forms of the risk aversion coefficients (3) and the parametric portfolio policy rule (6). The system of equations defined in (13) entails the existence of testable restrictions implied by the econometric model. Estimation of μ sets to zero $mn + 2n$ linear combinations of the mn^2 sample orthogonality conditions $g_T(\mu, \omega)$ with $\omega \in [\omega_{\min}, \omega_{\max}]$. The correct specification of the model implies that, for a fixed ω_0 , there are $mn^2 - mn - 2n$ linearly independent combinations of $g_T(\widehat{\mu}_T, \omega_0)$ that should be close to zero but are not exactly equal to zero. This hypothesis is tested using the Hansen test statistic (Hansen, 1982).

Let $s(\widehat{\mu}_T, \omega_0) = g_T(\widehat{\mu}_T, \omega_0)' \widehat{V}_T^{-1}(\omega_0) g_T(\widehat{\mu}_T, \omega_0)$, that under the null hypothesis of correct specification of the model, satisfies

$$s(\widehat{\mu}_T, \omega_0) \xrightarrow{d} \chi_{mn^2 - mn - 2n}^2. \quad (30)$$

The null hypothesis of correct specification of the overidentified equations is rejected at a significance level α if the test statistic $s(\widehat{\mu}_T, \omega_0)$ is greater than the critical value $\chi_{mn^2-mn-2n, 1-\alpha}^2$. In practice, the parameter ω_0 can be replaced by the estimator $\widehat{\omega}_T$ obtained from (18). A similar specification test can be developed to test the linear version of the above model against the model exhibiting constant risk aversion. In this case the relevant asymptotic condition is

$$s(\widehat{\mu}_{0T}) \xrightarrow{d} \chi_{mn^2-mn-n}^2, \quad (31)$$

with $s(\widehat{\mu}_{0T}) = g_T(\widehat{\mu}_{0T})' \widehat{V}_{0T}^{-1} g_T(\widehat{\mu}_{0T})$, where $g_T(\widehat{\mu}_{0T})$ and \widehat{V}_{0T} are the versions of the sample moment conditions and the empirical covariance function (21) obtained from the estimation of the linear dynamic model.

The second specification test that we discuss in this section allows us to compare different specifications of the multiperiod objective function (1) in terms of the individual's strategic horizon. To do this we devise a Wald type test that assesses the suitability of specific choices of the strategic horizon against alternative specifications. This test can naturally accommodate truncations of the infinite horizon model. Let us consider two different strategic horizons K_1 and K_2 such that $K_1 = K^* < K_2 \leq \infty$. The motivation for our test is to assume that for a given K^* the contribution of $\delta_{t,j}^j \psi_{t,j}(z_s; \lambda_h, \mu, \eta, \omega_0)$ $z_{\tilde{s},t}$ is not statistically significant for $j > K^*$. Let $\nu(\omega_0) = E \left[\sum_{j=K_1+1}^{K_2} \delta_{t,j}^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega_0) z_{\tilde{s},t} \right]$ denote a mn^2 vector of moment conditions indexed by $h = 1, \dots, m$ and $s, \tilde{s} = 1, \dots, n$. Under the null hypothesis,

$$H_{0,\nu} : \nu_1(\omega_0) = \dots = \nu_{mn^2}(\omega_0) = 0 \text{ against } H_{A,\nu} : \nu_s(\omega_0) \neq 0 \text{ for some } s = 1, \dots, mn^2. \quad (32)$$

Let $\widehat{\nu}_T(\omega_0) = \frac{1}{T-K_2} \sum_{t=1}^{T-K_2} \widehat{e}_{hs,t}(\omega_0) z_{\tilde{s},t}$ with $\widehat{e}_{hs,t}(\omega_0) = \sum_{j=K_1+1}^{K_2} \widehat{\delta}_{t,j}^j \psi_{t,j}(z_s; \widehat{\lambda}_h, \widehat{\gamma}_T, \widehat{\eta}_T, \omega_0)$ be the sample counterpart of the vector $\nu(\omega_0)$ indexed by $h = 1, \dots, m$ and $s, \tilde{s} = 1, \dots, n$. The estimator $\widehat{\mu}_T = (\widehat{\lambda}_T, \widehat{\gamma}_T, \widehat{\eta}_T)$ of μ is obtained under the alternative hypothesis characterized by

a longer investment horizon in the individual's objective function. Under the null hypothesis $H_{0,\nu}$ the consistency of the sample moments to $\nu(\omega_0)$ entails that $\widehat{\nu}_T(\omega_0) \xrightarrow{p} 0$, and a suitable test statistic for the joint hypothesis $H_{0,\nu}$ is the Wald test

$$W_r(\widehat{\mu}_T, \omega_0) = (T - K_2)\widehat{\nu}'_T \widetilde{V}_T^{-1}(\omega_0)\widehat{\nu}_T \quad (33)$$

with

$$\widetilde{V}_T(\omega_0) = \frac{1}{T - K_2} \sum_{t=1}^{T-K_2} \widetilde{e}_{h_1 s_1, t}(\omega_0) \widetilde{e}_{h_2 s_2, t}(\omega_0) z_{\widetilde{s}_1, t} z_{\widetilde{s}_2, t} + \frac{1}{T - K_2} \sum_{t=1}^{T-K_2} \sum_{\substack{t'=1 \\ t' \neq t}}^{T-K_2} \widetilde{e}_{h_1 s_1, t}(\omega_0) \widetilde{e}_{h_2 s_2, t'}(\omega_0) z_{\widetilde{s}_1, t} z_{\widetilde{s}_2, t'}, \quad (34)$$

that under the null hypothesis $H_{0,\nu}$, it satisfies

$$W_r(\widehat{\mu}_T, \omega_0) \xrightarrow{d} \chi_{mn}^2. \quad (35)$$

The alternative hypothesis implies the rejection of the truncation of the individual's strategic objective function by the first K^* periods. In practice, the parameter ω_0 can be replaced by the estimator $\widehat{\omega}_T$ obtained from (18).

5 Empirical application

In this section we analyze the optimal portfolio decisions and risk aversion dynamics of an strategic individual with objective function characterized by the time preference parameter $\beta = 0.95$, an strategic horizon of one year ($K = 12$) and a parameter $\theta = 0.999$ that sets individual's consumption to be a tiny fraction of the wealth in each period¹. Our aim is to compare the

¹The results are qualitatively similar for $\theta \in [0.90, 0.999]$, however, for values of θ smaller than 0.90 the algorithm is very unstable and does not converge in many cases.

optimal portfolio choices and risk aversion attitudes of three different types of strategic investors: individuals that exhibit a constant relative risk aversion coefficient, individuals that exhibit a risk aversion coefficient that varies over time according to the dynamics of our set of state variables, and finally, individuals with a dynamic nonlinear relative risk aversion coefficient.

We consider a tactical asset allocation setting characterized by a portfolio of stocks, bonds and the one-month real Treasury bill rate. As in Campbell et al. (2003), we do not impose short-selling restrictions. Our data covers the period January 1980 to December 2010. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index. The G0Q0 Bond Index is a Bank of America and Merrill Lynch U.S. Treasury Index that tracks the performance of U.S. dollar denominated sovereign debt publicly issued by the U.S. government in its domestic market. The nominal yield on the U.S. one-month risk-free rate is obtained from the Fama and French database, and the consumer price index (CPI) time series and the yield of the Moody's Baa- and Aaa-rated corporate bonds from the U.S. Federal Reserve.

The time-variation of the investment opportunity set is described by a set of state variables that have been identified in the empirical literature as potential predictors of the excess stock and bond returns and the short-term ex-post real interest rates. These variables are the detrended short-term interest rate (Campbell, 1991), the U.S. credit spread (Fama and French, 1989), the S&P 500 trend (Aït-Sahalia and Brandt, 2001) and the one-month average of the excess stock and bond returns (Campbell et al., 2003). The detrended short-term interest rate detrends the short-term rate by subtracting a 12-month backwards moving average. The U.S. credit spread is defined as the yield difference between Moody's Baa- and Aaa-rated corporate bonds. The S&P 500 momentum is the difference between the log of the current S&P 500 index level and the average index level over the previous 12 months. We demean and standardize all the state variables in the optimization process (Brandt et al, 2009).

Table 1 reports the sample statistics of the annualized excess stock return, excess bond

return and short-term ex-post real interest rates. The bond market outperforms the stock market during this period. In particular, the excess return on the bond index is higher than for the S&P 500 and exhibits a lower volatility entailing a Sharpe ratio almost three times higher for bonds than stocks. Additionally, the excess bond return has larger skewness and lower kurtosis.

[Insert Table 1 about here]

5.1 Empirical results

The parameter estimates driving the optimal portfolio rules and dynamic risk aversion coefficients are estimated using a two-step Gauss-Newton type algorithm using numerical derivatives and is implemented in Matlab. In a first stage we initialize the covariance matrix \widehat{V}_T with the matrix $I_{mn} \otimes Z'Z$, and in a second stage, after obtaining a first set of parameter estimates, we repeat the estimation replacing this matrix by a trimmed version of (17). In particular, we use a Newey-West estimator of the matrices $V_0(\omega)$ with $K = 12$ lags for different choices of ω within the compact set. The covariance matrix $\widehat{V}_T(\omega)$ is also used to perform the different threshold nonlinearity and specification tests described below.

Table 2 reports estimates of the model parameters for the three models under investigation. The first column contains the estimates of the nonlinear process characterized by a threshold nonlinearity on the risk aversion coefficient. The second column reports the parameter estimates of a simplified version of this model characterized by linear dynamics in the risk aversion coefficient. The third column contains the benchmark static model employed in the literature. Note that in contrast to most of the related literature that exogenously imposes different values of γ and compares the results across specifications, our method allows us to estimate the dynamics of risk aversion along with the parameters driving the portfolio weights from the data. Table 2 uncovers three important findings. First, the optimal portfolio weights are driven

by the dynamics of the state variables. Second, the risk aversion coefficient is dynamic and responds to the evolution of z_t ; Third, the nonlinear model adds flexibility in the modeling of the risk aversion coefficient compared to the linear model. The two-step estimation procedure establishes the presence of a structural break in the function $\gamma(j)$ at the seventh lag ($\widehat{k}_T = 7$) of the individual's strategic horizon. The objective function (16) takes the values 0.0569, 0.0541, 0.0540, 0.0539, 0.0536, 0.0509, 0.0524, 0.0544, and 0.0550 for $k = 2, 3, \dots, 10$, respectively. This result implies the existence of an interior solution to the minimization problem (18), and provides statistical support to the presence of two regimes in the risk aversion coefficient. The short-term regime is given by $\gamma'z_t$ and corresponds to the relative risk aversion coefficient exhibited by the individual over the first seven investment horizons of the multiperiod utility function. The long-term regime, characterized by $(\gamma + \eta)'z_t$, reflects relative risk aversion for the second period.

We proceed to explain these findings in more detail. The magnitude of the parameter estimates in table 2 shows the relevance of the state variables in describing the optimal portfolio weights and also the risk aversion coefficients. The statistical significance of these parameters is particularly strong for the parameters driving the portfolio weights but the state variables also exhibit predictive power for the linear and nonlinear characterizations of the risk aversion function $\gamma(j)$. The magnitude of the parameters driving the dynamics of risk aversion for both the linear and nonlinear models is large. These results are further supported by the different likelihood ratio tests comparing the model with constant risk aversion against the linear dynamic model, and the latter model against the model with a threshold nonlinearity in the investment plan. In particular, the value of the test statistic (24) is 11.93 that yields a p-value of 0.018. The p-value of the nonlinearity likelihood ratio test is computed using the algorithm described in the preceding section considering $J = 1000$ and a partition of the threshold variable given by $\Omega_N = 2, \dots, 10$. The test statistic of the test (20) is very high and the corresponding p-value is

zero. Figure 1 displays a rough nonparametric estimate of the density function of the supremum of the simulated chi-square process (29). The importance of the state variables in determining the individual's optimal portfolio decisions is mixed and depends on the specific asset of the portfolio and the model fitted to the data. Thus, most of the state variables exhibit statistical power to describe the dynamic allocation to bonds in the portfolio, however, this is not the case for the optimal allocation to stocks. In particular, the one-month average of the excess stock and bond returns is the only state variable with power to predict changes in the optimal allocation to the S&P 500 index.

[Insert Table 2 about here]

The interpretation of the parameters driving the dynamics of risk aversion also provides interesting observations. First, the risk aversion coefficient estimate $\hat{\gamma}_c$ is very significant and takes values between 3.195 and 3.491. The predictive power of the state variables for the dynamics of risk aversion is weak as the p-values of the marginal t-tests reveal². The nonlinear model is determined by a threshold value of $\hat{k}_T = 7$. The statistical significance of the models with dynamic risk aversion is empirically supported by the specification tests (30) and (31). The p-values of the tests oscillate between 0.90 and 0.97 for the three alternative specifications characterized by different risk aversion functions validating the parametric formulation of the optimal portfolio weights in (6) and the different specifications of the risk aversion functions in (3).

To illustrate the dynamics of the risk aversion coefficient we report in figure 2 the functions $\bar{\gamma}$, $\gamma(j)$ and γ_c , with $\bar{\gamma} = \left(\hat{k}_T \gamma(\hat{k}_T) + (K - \hat{k}_T) \gamma(\hat{k}_T + 1) \right) / K$ the cross-sectional average of the risk aversion coefficients across the investment horizon. Note that this function is determined

²Unreported exercises using a convergence criteria of $1e - 07$ in the minimization of the GMM algorithm report p-values smaller than 0.05. Nevertheless, for the sake of comparison across specifications of the objective function (1), and to save computational time, we have chosen a convergence criteria given by $1e - 05$ across iterations.

by two regimes given by $\gamma(1) = \dots = \gamma(\widehat{k}_T) = \gamma' z_t$ and $\gamma(\widehat{k}_T + 1) = \dots = \gamma(K) = (\gamma + \eta)' z_t$ denoting the short and long term strategic horizons, respectively; γ_c denotes the risk aversion coefficient obtained from the CRRA model exhibiting constant risk aversion. The unconditional risk aversion coefficient obtained from the latter model is $\exp(\widehat{\gamma}_c) = 33$ and is represented by a flat line. The functions $\bar{\gamma}$ (dashed line) and $\gamma(j)$ (dotted line) exhibit fluctuations around the flat line after the first years of data. During this period the chart also reveals a larger degree of risk aversion for the dynamic models than for the static model. This trend is compensated during the period 2000 – 2006 corresponding to the Great Moderation. During this episode the dynamic risk aversion coefficient is below the coefficient γ_c .

[Insert Figures 1-3 about here]

Figure 3 presents the separate dynamics of the short term $\gamma(\widehat{k}_T)$ and long term $\gamma(\widehat{k}_T + 1)$ risk aversion functions. The solid line corresponds to short term risk aversion and the dashed line to long term risk aversion. The solid line is stable and oscillates around a value of 33. The dashed line shows, instead, significant variation reaching values well above the short term risk aversion coefficients. This phenomenon is particularly relevant during periods corresponding to crisis episodes such as in the decade of 1980 or the 2007 – 2010 period. This finding highlights the nonlinear behavior of risk averse individuals. The increase in risk aversion due to adverse market events is mostly reflected in long term risk aversion.

5.2 Robustness analysis

In this section we perform two robustness exercises that assess the consistency of the above results over other environments. First, we entertain an objective function with $K = 24$ periods, and second, we study the implications of the multiperiod utility function for an investor only concerned with maximizing the utility of wealth ($\theta = 1$).

Table 3 reports the parameter estimates of the version of model (1) with two years in the individual’s strategic horizon. The results are similar in spirit to the findings previously obtained for $K = 12$, however, for a longer investment plan the state variables become better predictors, and statistically more significant, of the dynamics of the optimal portfolio weights. More formally, in contrast to the previous case, the detrended short-term interest rate also becomes a statistically significant predictor of the allocation to stocks. The optimal allocation to bonds still benefits from the predictive power of all of the state variables proposed in this application.

[Insert Table 3 about here]

The specification tests also report p-values near one providing empirical support to the different specifications of the individual’s multiperiod objective function. The nonstandard likelihood ratio test reports a value greater than one thousand and hence a p-value of zero favouring the nonlinear model against the linear model. Figure 4 presents a nonparametric density function of the associated $\chi^2(\omega)$ process. The likelihood ratio test (20) reports a value of 91.46 that yields a zero p-value and supports the dynamic linear model against the model with constant relative risk aversion.

Figure 5 reveals more significant differences between the average nonlinear risk aversion function $\bar{\gamma}$ and the linear function $\gamma(j)$ than for $K = 12$. This empirical finding is also noted in figure 6. This graph shows a significantly higher risk aversion to the long term given by the function $\gamma(8)$ than to the short term characterized by $\gamma(7)$. Interestingly, we find the same threshold value $\hat{k}_T = 7$ as in the model with twelve periods in the individual’s strategic horizon. The similarities in the nonlinear function across specifications of the strategic horizon provide further support to the existence of a threshold nonlinearity in the individual’s objective function. This observation suggests that as the number of periods in the individual’s strategic

horizon increases the long term risk aversion component becomes more relevant and leads to more conservative behaviors.

[Insert Figures 4 - 6 about here]

The second robustness exercise analyses the sensitivity of the results to considering the optimal choices of an individual obtaining utility from wealth rather than consumption. More specifically, table 4 reports the parameter estimates driving the optimal portfolio weights and the different dynamic risk aversion functions for $\theta = 1$ and $K = 12$. The results are very similar to the findings obtained in tables 2 and 3. The threshold nonlinearity in the risk aversion function is also obtained at $\hat{k}_T = 7$. The magnitude of the parameter estimates is similar across specifications, however, the predictive power of the state variables is statistically much more significant than in previous specifications based on consumption. Similarly, we observe that the state variables exhibit stronger power to predict risk aversion than before. This is mainly reflected in the parameter estimates of the coefficients of risk aversion in the linear dynamic case and by the different likelihood ratio tests. The test statistic of the nonlinear likelihood ratio test reports a value of 211 that yields a p-value of zero. Figure 7 reports a nonparametric estimator of the density function of the supremum of the simulated $\chi^2(\omega)$ process under the null hypothesis. Similarly, the likelihood ratio test between the linear dynamic model and the model with constant risk aversion reports a test statistic of 14.89 that yields a p-value of 0.005, leading us to conclude that the individual's relative risk aversion coefficient is dynamic.

[Insert Table 4 about here]

In the three specifications the estimates of the risk aversion coefficient γ_c are greater than in the models characterized by utility over consumption. The values oscillate in this case between 4.121 and 4.214 yielding risk aversion coefficients between 62 and 68. Individuals

become more risk averse when their preferences build over wealth than over consumption. Figures 8 and 9 report the dynamics of the risk aversion functions in this scenario. The risk aversion functions $\bar{\gamma}$ and $\gamma(j)$ show very similar patterns. Risk aversion to the long term is still higher than to the short term but the premium is small compared to previous specifications. Interestingly, in contrast to the previous specifications characterized by individual's preferences over consumption, we observe in this scenario that periods of financial distress trigger increases in short term risk aversion but not in long term risk aversion.

[Insert Figures 7 - 9 about here]

6 Conclusion

This paper studies the influence of assuming dynamic risk aversion in the optimal asset allocation of strategic individuals concerned with maximizing the utility of their stream of consumption or wealth over multiple, potentially infinite, periods. To do this we have developed a theoretical and empirical framework in an economy populated by individuals characterized by three distinguishing features. First, individuals behave strategically in the sense that their optimal decisions are far sighted as they also obtain utility from periods further into the future. Second, individuals' relative risk aversion reacts to macroeconomic and financial conditions proxied by a set of state variables. Third, the model accommodates nonlinearities in the risk aversion function that are interpreted as evidence of heterogeneity in the individual's risk aversion coefficient between the short and long term.

The parameters defining this model are estimated using GMM procedures applied to an overidentified system of equations describing the first order conditions of the individual's multi-period maximization problem. The overidentification property allows us to test the suitability of the parametric specifications defining our model specification: a linear portfolio policy rule

for the dynamics of the portfolio weights and a threshold piecewise linear specification for the relative risk aversion coefficient.

The empirical application to a tactical portfolio of three assets - a one-month Treasury bill as riskless security, a long-term bond, and an equity portfolio finds overwhelming empirical evidence of the presence of dynamics and nonlinearities in the risk aversion coefficient. The different dynamics reported by the two segments of the risk aversion function are interpreted as risk aversion to the short term and long term, respectively. The state variables proposed in this paper to describe the dynamics of the optimal portfolio weights exhibit more predictive power, and hence, more exposure to the allocation to bonds than stocks. All of the state variables proposed in our study have statistical relevance to explain the dynamic allocation to bonds, however, the allocation to stocks is mainly driven by the one-month average of the excess stock and bond returns. Similarly, our model reveals that the dynamics of the risk aversion coefficient are determined by the U.S. credit spread, defined as the yield difference between Moody's Baa- and Aaa-rated corporate bonds and the S&P 500 momentum, defined as the difference between the log of the current S&P 500 index level and the average index level over the previous 12 months.

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Tables and figures

asset	Mean	Volatility	Sharpe ratio	Skewness	Kurtosis
S&P500 index	0.027	0.131	0.210	-1.120	4.88
G0Q0 Bond index	0.029	0.056	0.510	0.151	2.170
r_f	0.018	0.021	–	0.380	3.16

Table 1. Summary statistics of the excess stock return, excess bond return and short-term ex-post real interest rates over the period January 1980 to December 2010. The return horizon is one month. Mean and volatility are expressed in annualized terms.

stock parameters				bond parameters			
θ	nonlinear	dynamics	constant	θ	nonlinear	dynamics	constant
$\lambda_{s,c}$	0.177 [0.029]	0.166 [0.011]	0.182 [0.007]	$\lambda_{b,c}$	0.393 [0.032]	0.343 [0.028]	0.303 [0.026]
$\lambda_{s,1}$	0.081 [0.162]	0.038 [0.267]	0.036 [0.249]	$\lambda_{b,1}$	-0.115 [0.138]	-0.156 [0.036]	-0.180 [0.016]
$\lambda_{s,2}$	-0.006 [0.899]	0.032 [0.356]	0.041 [0.145]	$\lambda_{b,2}$	-0.696 [0.014]	-0.638 [0.006]	-0.624 [0.005]
$\lambda_{s,3}$	-0.032 [0.560]	-0.024 [0.493]	-0.034 [0.294]	$\lambda_{b,3}$	-0.608 [0.018]	-0.516 [0.010]	-0.511 [0.008]
$\lambda_{s,4}$	0.456 [0.013]	0.421 [0.008]	0.488 [0.006]	$\lambda_{b,4}$	-0.251 [0.058]	-0.235 [0.026]	-0.207 [0.024]

short term regime				long term regime			
θ	nonlinear	dynamics	constant	θ	nonlinear	dynamics	constant
γ_c	3.195 [0.000]	3.491 [0.000]	3.467 [0.000]	η_c	0.365 [0.307]	-	-
γ_1	0.082 [0.484]	-0.004 [0.961]	-	η_1	-0.196 [0.571]	-	-
γ_2	0.061 [0.732]	0.097 [0.221]	-	η_2	0.352 [0.543]	-	-
γ_3	0.030 [0.825]	0.101 [0.148]	-	η_3	0.248 [0.453]	-	-
γ_4	0.343 [0.101]	0.182 [0.289]	-	η_4	-0.433 [0.314]	-	-

Table 2. Parameter estimates of the three different versions of the individual's objective function (1). The model is completed by assuming $K = 12$, $\theta = 0.999$ and $\beta = 0.95$. The state variables z_t are the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of the excess stock and bond returns.

stock parameters				bond parameters			
θ	nonlinear	dynamics	constant	θ	nonlinear	dynamics	constant
$\lambda_{s,c}$	0.198 [0.087]	0.205 [0.000]	0.260 [0.000]	$\lambda_{b,c}$	0.598 [0.053]	0.349 [0.005]	0.415 [0.000]
$\lambda_{s,1}$	0.061 [0.293]	0.083 [0.011]	0.049 [0.146]	$\lambda_{b,1}$	-0.246 [0.066]	-0.266 [0.000]	-0.323 [0.000]
$\lambda_{s,2}$	-0.048 [0.450]	-0.027 [0.474]	-0.039 [0.233]	$\lambda_{b,2}$	-0.961 [0.031]	-0.809 [0.000]	-0.879 [0.000]
$\lambda_{s,3}$	0.047 [0.426]	0.001 [0.972]	0.014 [0.611]	$\lambda_{b,3}$	-0.804 [0.047]	-0.697 [0.006]	-0.701 [0.003]
$\lambda_{s,4}$	0.616 [0.030]	0.615 [0.000]	0.797 [0.000]	$\lambda_{b,4}$	-0.411 [0.041]	-0.305 [0.000]	-0.282 [0.000]
short term regime				long term regime			
θ	nonlinear	dynamics	constant	θ	nonlinear	dynamics	constant
γ_c	2.938 [0.000]	3.195 [0.000]	3.033 [0.000]	η_c	0.668 [0.106]	-	-
γ_1	-0.041 [0.669]	-0.077 [0.259]	-	η_1	-0.309 [0.087]	-	-
γ_2	0.241 [0.040]	0.053 [0.278]	-	η_2	0.114 [0.777]	-	-
γ_3	0.254 [0.032]	0.236 [0.000]	-	η_3	0.414 [0.170]	-	-
γ_4	0.209 [0.321]	0.018 [0.908]	-	η_4	-0.254 [0.559]	-	-

Table 3. Parameter estimates of the three different versions of the individual's objective function (1). The model is completed by assuming $K = 24$, $\theta = 0.999$ and $\beta = 0.95$. The state variables z_t are the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of the excess stock and bond returns.

stock parameters				bond parameters			
θ	nonlinear	dynamics	constant	θ	nonlinear	dynamics	constant
$\lambda_{s,c}$	0.068 [0.014]	0.075 [0.004]	0.084 [0.000]	$\lambda_{b,c}$	0.165 [0.006]	0.161 [0.004]	0.146 [0.002]
$\lambda_{s,1}$	0.040 [0.085]	0.036 [0.059]	0.033 [0.043]	$\lambda_{b,1}$	-0.054 [0.182]	-0.045 [0.217]	-0.064 [0.033]
$\lambda_{s,2}$	0.024 [0.185]	0.032 [0.068]	0.033 [0.015]	$\lambda_{b,2}$	-0.313 [0.000]	-0.322 [0.000]	-0.325 [0.000]
$\lambda_{s,3}$	-0.007 [0.703]	-0.006 [0.661]	-0.012 [0.391]	$\lambda_{b,3}$	-0.240 [0.000]	-0.243 [0.000]	-0.241 [0.000]
$\lambda_{s,4}$	0.221 [0.000]	0.217 [0.000]	0.246 [0.000]	$\lambda_{b,4}$	-0.099 [0.059]	-0.094 [0.041]	-0.091 [0.026]
short term regime				long term regime			
θ	nonlinear	dynamics	constant	θ	nonlinear	dynamics	constant
γ_c	4.121 [0.000]	4.214 [0.000]	4.171 [0.000]	η_c	0.170 [0.619]	-	-
γ_1	-0.002 [0.979]	0.007 [0.924]	-	η_1	-0.043 [0.882]	-	-
γ_2	0.185 [0.318]	0.110 [0.096]	-	η_2	-0.016 [0.973]	-	-
γ_3	0.110 [0.318]	0.126 [0.064]	-	η_3	0.105 [0.637]	-	-
γ_4	0.288 [0.123]	0.094 [0.653]	-	η_4	-0.338 [0.448]	-	-

Table 4. Parameter estimates of the three different versions of the individual's objective function (1). The model is completed by assuming $K = 12$, $\theta = 1$ and $\beta = 0.95$. The state variables z_t are the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of the excess stock and bond returns.

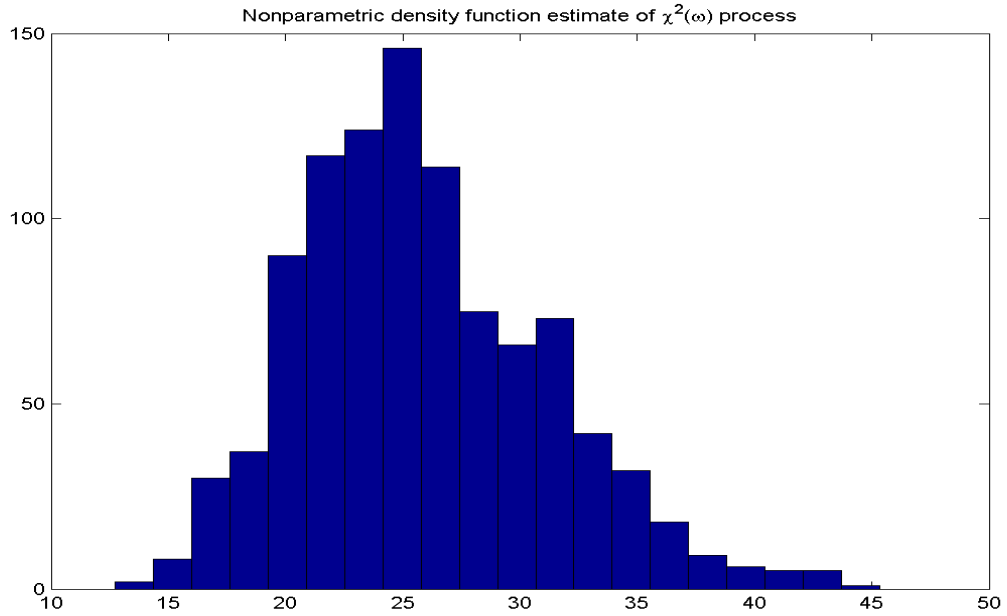


Figure 1. Nonparametric density function for the likelihood ratio test (20) for the case when ω is not observed. The asymptotic distribution of the $\chi^2(\omega)$ process (26) is approximated using the p-value transformation discussed in the paper; $J = 1000$ and a partition of $[0, 1]$ given by $\Omega_N = 2, \dots, 10$. The objective function (1) considers $K = 12$ and $c_t = (1 - \theta)w_t$ with $\theta = 0.999$.

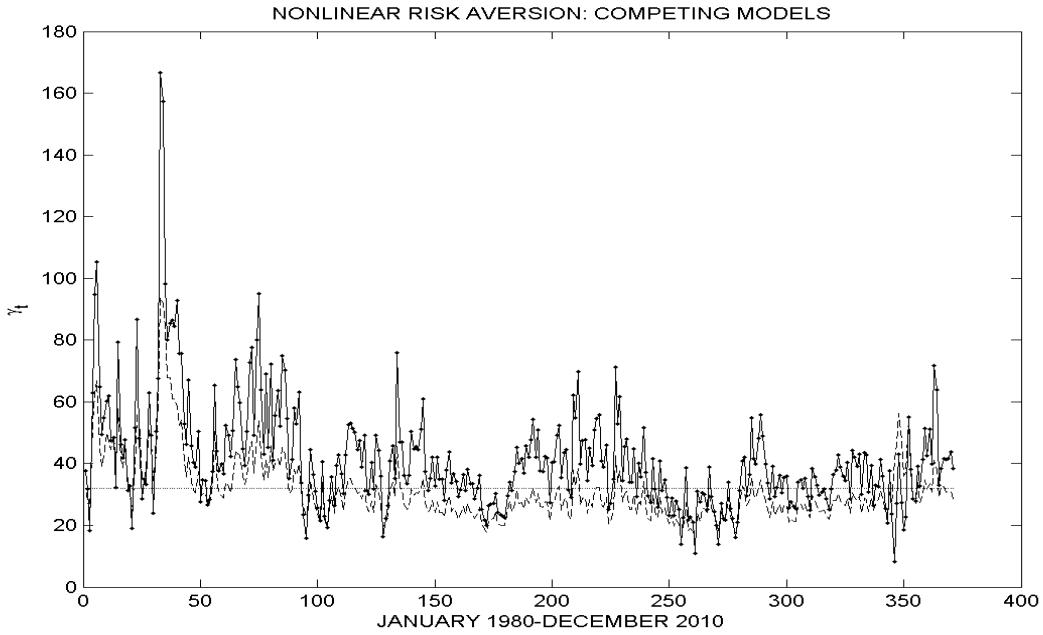


Figure 2. Dynamics of risk aversion over the sample period January 1980 to December 2010. The flat line corresponds to the model with constant risk aversion. The dashed line corresponds to the nonlinear threshold model and the dotted line corresponds to the model with linear dynamics. The time series of risk aversion for the nonlinear model is constructed as a weighted average of the short term and long term risk aversion coefficients. The objective function (1) considers $K = 12$ and $c_t = (1 - \theta)w_t$ with $\theta = 0.999$.

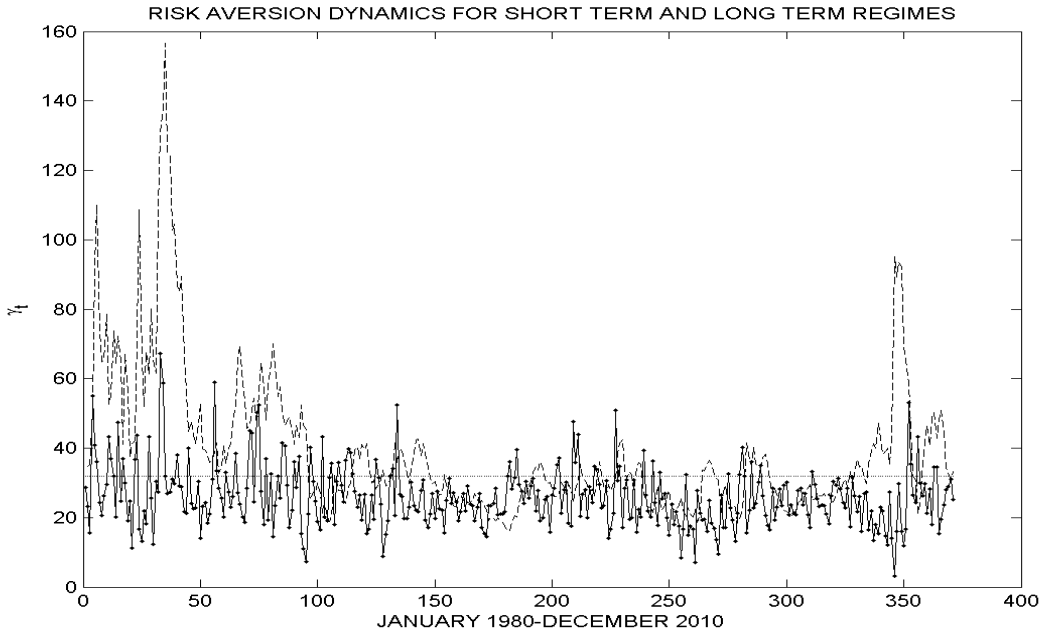


Figure 3. Dynamics of risk aversion over the sample period January 1980 to December 2010. The solid line corresponds to the dynamics of short term risk aversion characterized by the first seven periods of the individual's investment horizon. The dashed line corresponds to the dynamics of long term risk aversion. The flat line corresponds to the model with constant risk aversion. The objective function (1) considers $K = 12$ and $c_t = (1 - \theta)w_t$ with $\theta = 0.999$.

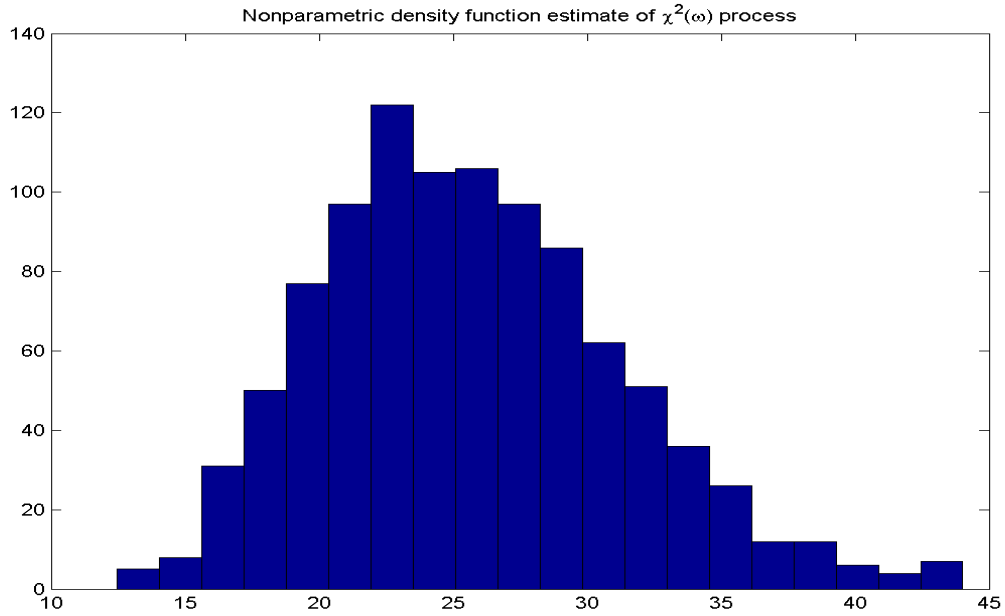


Figure 4. Nonparametric density function for the likelihood ratio test (20) for the case when ω is not observed. The asymptotic distribution of the $\chi^2(\omega)$ process (26) is approximated using the p-value transformation discussed in the paper; $J = 1000$ and a partition of $[0, 1]$ given by $\Omega_N = 2, \dots, 20$. The objective function (1) considers $K = 24$ and $c_t = (1 - \theta)w_t$ with $\theta = 0.999$.

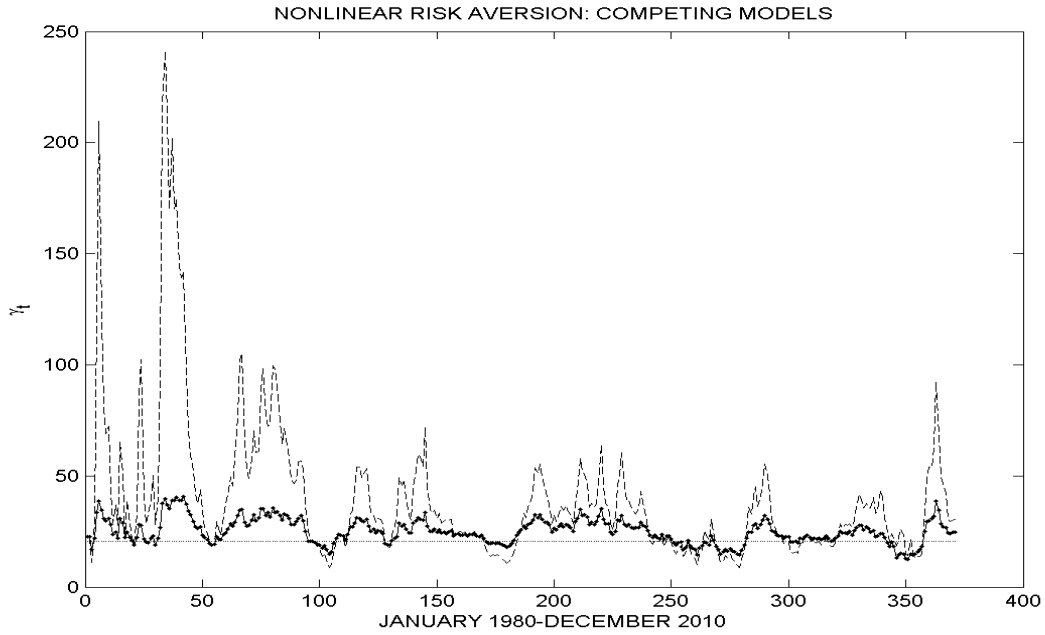


Figure 5. Dynamics of risk aversion over the sample period January 1980 to December 2010. The flat line corresponds to the model with constant risk aversion. The dashed line corresponds to the nonlinear threshold model and the dotted line corresponds to the model with linear dynamics. The time series of risk aversion for the nonlinear model is constructed as a weighted average of the short term and long term risk aversion coefficients. The objective function (1) considers $K = 24$ and $c_t = (1 - \theta)w_t$ with $\theta = 0.999$.

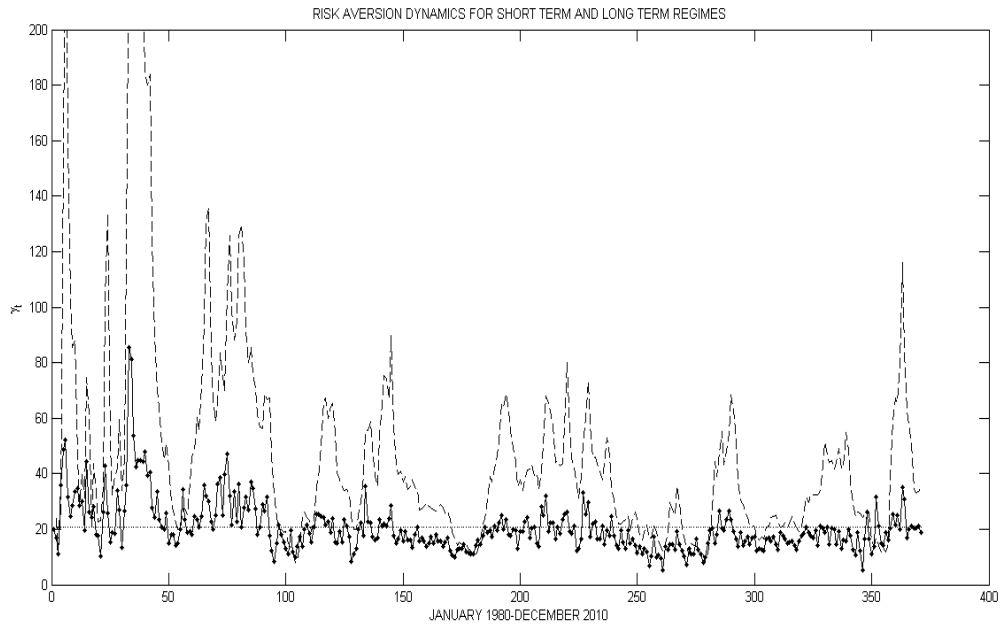


Figure 6. Dynamics of risk aversion over the sample period January 1980 to December 2010. The solid line corresponds to the dynamics of short term risk aversion characterized by the first seven periods of the individual's investment horizon. The dashed line corresponds to the dynamics of long term risk aversion. The flat line corresponds to the model with constant risk aversion. The objective function (1) considers $K = 24$ and $c_t = (1 - \theta)w_t$ with $\theta = 0.999$.

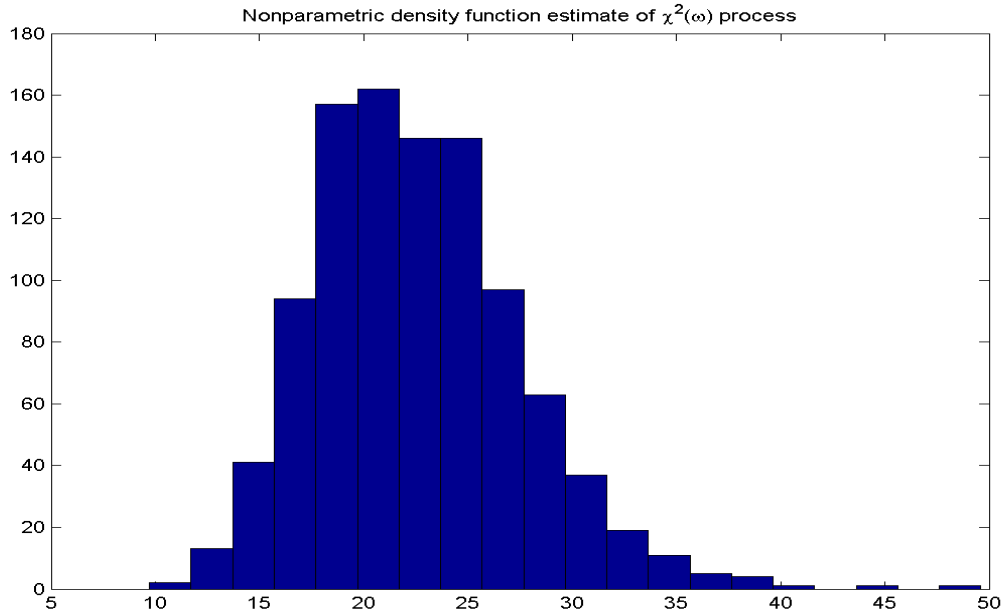


Figure 7. Nonparametric density function for the likelihood ratio test (20) for the case when ω is not observed. The asymptotic distribution of the $\chi^2(\omega)$ process (26) is approximated using the p-value transformation discussed in the paper; $J = 1000$ and a partition of $[0, 1]$ given by $\Omega_N = 2, \dots, 10$. The objective function (1) considers $K = 12$ and no consumption.

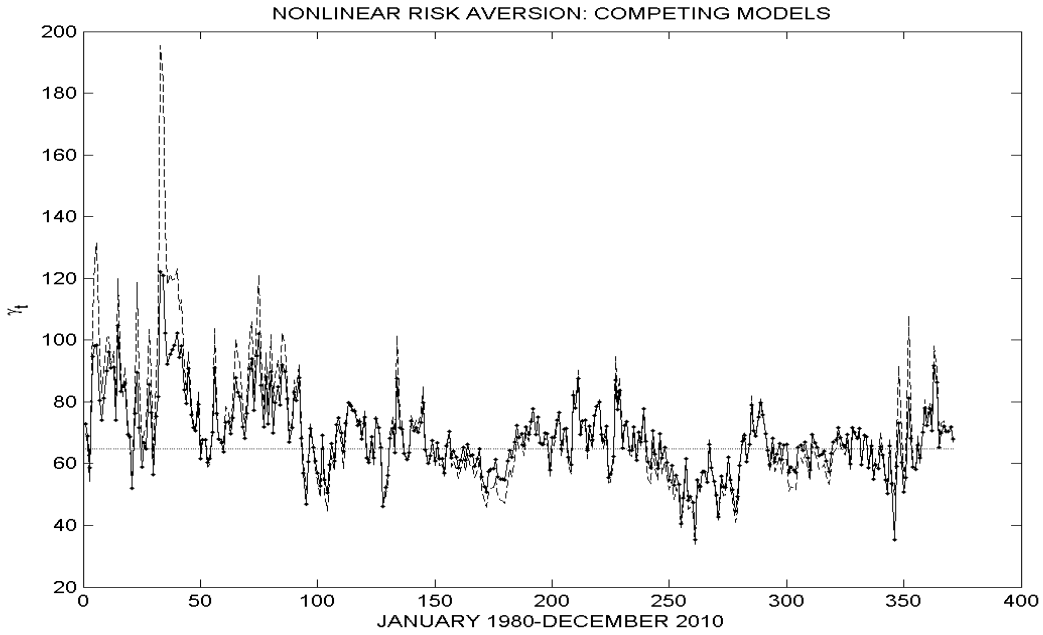


Figure 8. Dynamics of risk aversion over the sample period January 1980 to December 2010. The flat line corresponds to the model with constant risk aversion. The dashed line corresponds to the nonlinear threshold model and the dotted line corresponds to the model with linear dynamics. The time series of risk aversion for the nonlinear model is constructed as a weighted average of the short term and long term risk aversion coefficients. The objective function (1) considers $K = 12$ and no consumption.

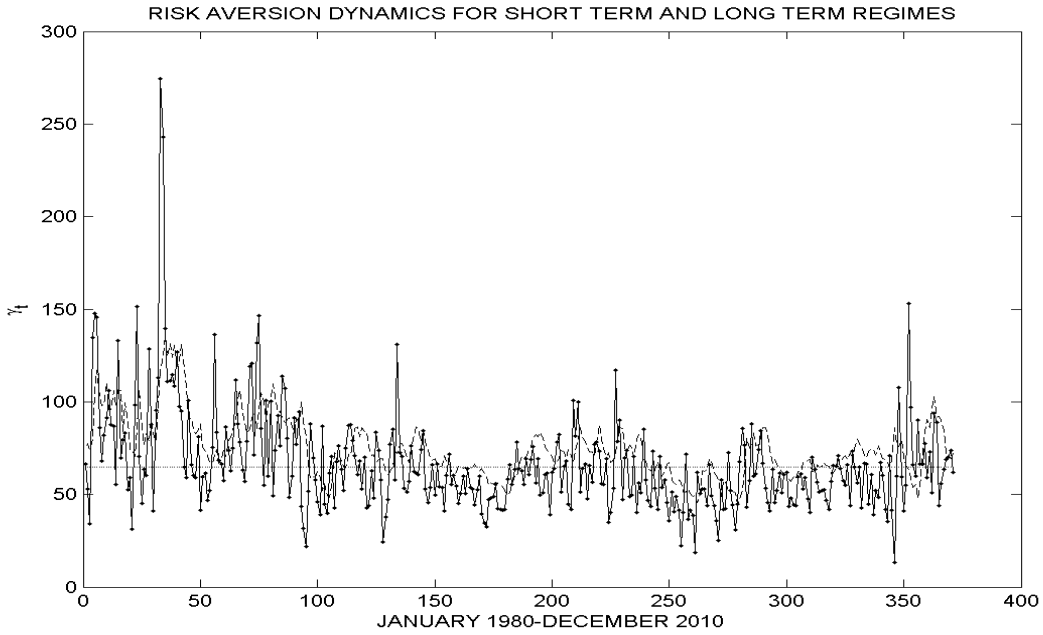


Figure 9. Dynamics of risk aversion over the sample period January 1980 to December 2010. The solid line corresponds to the dynamics of short term risk aversion characterized by the first seven periods of the individual's investment horizon. The dashed line corresponds to the dynamics of long term risk aversion. The flat line corresponds to the model with constant risk aversion. The objective function (1) considers $K = 12$ and no consumption.